Some Applications of Characteristic Roots and Characteristic Vectors

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Abstract

This paper deals with application to eigenvalue, eigenvector. Firstly the linear transformations of two vector spaces over the same field are studied. And then the characteristic roots and characteristics vectors of linear transformation are expressed. Finally, system of differential equations and orthogonal coordinate transformations to the quadratic equations are discussed.

Key words: linear transformation, invertible of algebra, eigenvalue, eigenvector

Introduction

A vector space *V* over the field *K* is a set of objects which can be added and multiplied by elements of *K*, and the eight properties are satisfied. A linear transformation is a mapping from vector space into vector space. Eigenvalues and eigenvectors of linear transformation are based on the algebra of linear transformation.

Linear Transformation of Vector Spaces

Definition (1)

Let V, V' be vector spaces over field *F*. A *linear transformation* $T: V \to V'$ is a mapping which satisfies the following two properties:

(i) For any elements u, v in V, we have

$$T(u+v) = T(u)+T(v)$$

(ii) For all $c \in F$ and v in V, we have

$$T(cv) = cT(v).$$

Invertible of Linear Transformation

A linear transformation $T: V \to W$ is *invertible* iff it is 1-1, onto and inverse of T is the mapping $T^{-1}: W \to V$ such that

$$T^{-1}(y) = x, \text{ then } T(x) = y.$$

Inverse of T is a linear transformation.

Indeed, since *T* is onto, for $w_1, w_2 \in W$, there exists $v_1, v_2 \in V$ such that

$$T(v_1) = w_1,\ T(v_2) = w_2 \Leftrightarrow v_1 = T^{-1}(w_1),\ v_2 = T^{-1}(w_2).$$

$$\begin{aligned} T^{-1}(\alpha w_1 + \beta w_2) &= T^{-1}(\alpha T(v_1) + \beta T(v_2)) \\ &= T^{-1}(T(\alpha v_1) + T(\beta v_2)) \\ &= T^{-1}(T(\alpha v_1 + \beta v_2)), \text{ Since T is linear.} \\ &= \alpha v_1 + \beta v_2 \\ &= \alpha T^{-1}(w_1) + \beta T^{-1}(w_2). \end{aligned}$$

Therefore T^{-1} is a linear transformation.

Definition (2)

The mapping $\phi: R \to R'$ of the ring *R* into the ring R' is a *homomorphsim* if

(i) $\phi(a+b) = \phi(a) + \phi(b)$ and

(ii) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R.$

Definition (3)

If ϕ is a homomorphism of *R* into R' then the *kernel* of ϕ, $\ker\phi$, is the set of all elements $a \in R$ such that $\phi(a) = 0'$, the zero element of R'.

Definition (4)

A linear transformation $T: V \to W$ is called *non-singular* if $\ker T = \{0\}$.

Theorem (1)

Let $T: V \to W$ be a linear transformation where *V* and *W* are two finite dimensional vector spaces with dimension. Then the following are equivalent:

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto (That is, Range of $T = W$).

(iv) If $\{v_1, v_2, \cdots, v_n\}$ is a basis of *V* then $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis of *W*.

The Algebra of Linear Transformations

Let V be a vector space over a field F and let $\mathrm{Hom}(V,V)$ the set of all vector space homomorphisms of V into itself. Then

(I) $T \in \mathrm{Hom}(V,V)$ and $T: V \to V$ by

(i) $(v_1 + v_2)T = v_1T + v_2T,$

(ii) $(\alpha v_1)T = \alpha(v_1T),$ for all $v_1, v_2 \in V, \alpha \in F.$

(II) (i) For $T_1, T_2 \in \mathrm{Hom}(V,V),$ $T_1 + T_2$ is defined by

$$v(T_1 + T_2) = vT_1 + vT_2, \text{ for all } v \in V.$$

(ii) For $T \in \mathrm{Hom}(V,V)$, $\alpha \in F, \alpha T$ is defined by

$$v(\alpha T) = \alpha(vT), \text{ for all } v \in V.$$

Then $\mathrm{Hom}(V,V)$ is a vector space homomorphism.

(III) For $T_1, T_2 \in \mathrm{Hom}(V,V),$ T_1T_2 is defined by

$$(au + \beta v)(T_1T_2) = \alpha(u(T_1T_2)) + \beta(v(T_1T_2)),$$

for all $u, v \in V$ and $\alpha, \beta \in F.$

Then $T_1T_2 \in \mathrm{Hom}(V,V).$

Definition (5)

An associative ring A is called an *algebra* over F, if A is a vector space over F such that for all $a, b \in A$ and $\alpha \in F, \alpha(ab) = (\alpha a)b = a(\alpha b).$

$\mathrm{Hom}(V,V)$ is an algebra over F, and we write

$$A(V) \text{ or } A_F(V).$$

Remark (1)

Let A be an algebra, with unit element, over F.

Let $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$ be a polynomial in $F[x],$ where $\alpha_0, \alpha_1, \cdots, \alpha_n \in F.$ For $a \in A, p(a) = \alpha_0 e + \alpha_1 a + \alpha_2 a^2 + \cdots + \alpha_n a^n$ is an element of A.

If $p(a) = 0,$ we say that a satisfies $p(x).$

Definition (6)

An element $T \in A(V)$ is called *right invertible*, if there exists $S \in A(V)$ such that $TS = I$ where I denotes the unit element in $A(V)$.

T is called *right invertible*, if there exists $U \in A(V)$ such that $UT = I.$

If T is both right and left invertible, we say T is *invertible*.

If $T \in A(V),$ then the range of T, VT, is defined by

$$VT = \{vT \mid v \in V\}.$$

Definition (7)

An element T in $A(V)$ is *invertible* or *regular* if it is both right and left invertible; that is, if there is an element $S \in A(V)$ such that $ST = TS = I.$ We write S as $T^{-1}.$

An element in $A(V)$ which is not regular is called *singular*.

An element in $A(V)$ is right invertible, but may not be invertible.

Corollary (1)

If V is finite dimensional over F and $T \in A(V)$ is singular, there exists an $S \neq 0$ in $A(V)$ such that $ST = TS = 0.$

Theorem (2)

If V is finite dimensional over F, then $T \in A(V)$ is singular if and only if there exists an $v \neq 0$ in V such that $vT = 0.$

Characteristic Roots of Linear Transformation

Definition (8)

Let V be finite dimensional over F. If $T \in A(V)$ then $\lambda \in F$ is called a *characteristic root* or *eigenvalue* of T if $\lambda - T$ is singular. That is, $\lambda - T = \lambda I - T.$

Theorem (3)

The element $\lambda \in F$ is a characteristic root of $T \in A(V)$ if and only if for some $v \neq 0$ in $V,$ $vT = \lambda v.$

Lemma (1)

If $\lambda \in F$ is a characteristic root of $T \in A(V),$ then for any polynomial $q[x] \in F[x], q(\lambda)$ is a characteristic root of $q[T].$

Characteristic Vector

Definition (9)

The element $v \neq 0$ in V is called a *characteristic vector* or *eigenvector* of T belonging to the characteristic root $\lambda \in F$ if $vT = \lambda v.$

Theorem (4)

If $\lambda_1, \lambda_2, \cdots, \lambda_k$ in F are distinct characteristic roots of $T \in A(V)$ and $v_1, v_2, \cdots, v_k$ are characteristic vectors of T belonging to $\lambda_1, \lambda_2, \cdots, \lambda_k$, respectively, then $v_1, v_2, \cdots, v_k$ are linearly independent over F.

Corollary (2)

If $T \in A(V)$ and $\dim_F V = n$ and if T has n distinct characteristic roots in F, then there is a basis of V over F which consists of characteristic vectors of T.

Example (1)

Let V be two-dimensional vector space over the field F, of real numbers, a basis v_1, v_2, then we can find the characteristic roots and corresponding characteristic vectors for T defined by

$$v_1T = v_1 + v_2, vT = v_1 - v_2.$$

Let λ be the characteristic root of T in F and v be characteristic vector corresponding to λ. By definition, there exists $v \neq 0$ in V such that $vT = \lambda v$. Since $\{v_1, v_2\}$ is a basis of V,

$v = av_1 + bv_2$ where $a, b \in F$ and not both zero.

$$vT = (av_1 + bv_2)T$$

$$= (a+b)v_1 + (a-b)v_2.$$

But $vT = \lambda v$. Therefore

$$\lambda v = (a+b)v_1 + (a-b)v_2$$

$$(a+b-\lambda a)v_1 + (a-b-\lambda b)v_2 = 0.$$

Since v_1, v_2 are linearly independent,

$a+b-\lambda a = 0, a-b-\lambda b = 0$ and hence

$(1-\lambda)a + b = 0, a-(1+\lambda)b = 0.$

For nontrivial solution for a and b,

$$\lambda = \pm\sqrt{2}, \text{ when } \lambda = \sqrt{2},$$

$$b = (\sqrt{2}-1)a.$$

Therefore $v = av_1 + bv_2 = av_1 + (\sqrt{2}-1)av_2$

$$= a[v_1 + (\sqrt{2}-1)v_2].$$

Put $a = 1$, we have $v = v_1 - (1+\sqrt{2})v_2$ is a characteristic vector corresponding to $\lambda = -\sqrt{2}$.

When $\lambda = -\sqrt{2},\ b = -(1-\sqrt{2})a.$

$$v = av_1 + bv_2 = av_1 - (1+\sqrt{2})av_2 = a[v_1 - (1+\sqrt{2})v_2].$$

Therefore, $v_1 + (\sqrt{2}-1)v_2$ and $v_1 - (1+\sqrt{2})v_2$ are characteristic vector for T.

Linear algebra can be applied to solve certain systems of differential equations. One of the simplest differential equations is

Application to Ordinary Differential Equations

$$y' = ay,$$

where $y = f(x)$ is an unknown function to be determined, $y' = \dfrac{dy}{dx}$ is its derivative, and a is a constant.

$$y = ce^{ax}$$

where c is an arbitrary constant. Each function of this form is a solution of $y' = ay$.

$$y' = cae^{ax} = ay.$$

Conversely, every solution of $y' = ay$ must be a function of the form ce^{ax}, so that describes all solutions of $y' = ay$. We call the general solution of $y' = ay$.

We will be concerned with solving systems of differential equations having the form:

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n, \end{aligned}$$

where $y_1 = f_1(x), y_2 = f_2(x), \cdots, y_n = f_n(x)$ are functions and a_{ij} 's are constants. In matrix notation can be written

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Y' = AY$$

where

$$Y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix},\ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},\ A = (a_{ij}),\ i, j = 1, 2, 3, \cdots, n.$$

Solution of the Linear System $Y' = AY$

We consider the system

$$Y' = AY$$

in which the matrix A is not diagonal.

Try to make a substitution for Y that will yield a new system with a diagonal coefficient matrix; solve this new simpler system, and then use this solution to determine the solution of the original system.

The kind of substitution we have in mind is

$$y_1 = p_{11}u_1 + p_{12}u_2 + \cdots + p_{1n}u_n$$
$$y_2 = p_{21}u_1 + p_{22}u_2 + \cdots + p_{2n}u_n$$
$$\vdots$$
$$y_n = p_{n1}u_1 + p_{n2}u_2 + \cdots + p_{nn}u_n$$

or in matrix notation

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$Y = PU.$$

In this substitution the p_{ij}'s are constants to be determined in such a way that the new system involving the unknown functions $u_1, u_2, \cdots, u_n$ has a diagonal coefficient matrix.

$$Y' = PU'.$$

If we make the substitutions $Y = PU$ and $Y' = PU'$ in the original system

$$Y' = AY$$

and if we assume P to be invertible, we obtain

$$PU' = A(PU)$$

$$U' = (P^{-1}AP)U$$

$$U' = DU, \text{ where } D = (P^{-1}AP).$$

Example (2)

Consider the system

$$y_1' = y_1 + y_2$$

$$y_2' = 4y_1 - 2y_2,$$

in the initial conditions $y_1(0) = 1, y_2(0) = 6.$

The coefficient matrix is

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Y' = AY, \; A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}.$$

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -1 \\ -4 & \lambda + 2 \end{bmatrix}.$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -4 & \lambda + 2 \end{vmatrix} = (\lambda + 3)(\lambda - 2).$$

The characteristic equation of A is

$$\det(\lambda I - A) = 0$$

$$\lambda = -3, \quad \lambda = 2.$$

The eigenvalues of A are $\lambda = -3, \quad \lambda = 2.$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda.$

$$(\lambda I - A)x = 0.$$

If $\lambda = 2,$

$$\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$x_1 - x_2 = 0$$

$$-4x_1 + 4x_2 = 0.$$

Let $x_1 = t, x_2 = t.$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore $p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 2.$

If $\lambda = -3,$

$$\begin{bmatrix} -4 & -1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$-4x_1 - x_2 = 0$$

$$-4x_1 - x_2 = 0.$$

Let $x_2 = t.$ Then $x_1 = -(1/4)t.$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(1/4)t \\ t \end{bmatrix} = t \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}.$$

Therefore $p_1 = \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = -3.$

Therefore $P = \begin{bmatrix} 1 & -1/4 \\ 1 & 1 \end{bmatrix}.$

$$D = P^{-1}AP$$
$$= \begin{bmatrix} 4/5 & -1/4 \\ -4/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1/4 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

$$Y = PU \text{ and } Y' = PU'.$$
$$U' = DU = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}U$$

or

$$u_1' = 2u_1$$
$$u_2' = -3u_2.$$

From $y = ce^{ax}$,

$$u_1 = c_1e^{2x}$$
$$u_2 = c_2e^{-3x}$$

or $U = \begin{bmatrix} c_1e^{2x} \\ c_2e^{-3x} \end{bmatrix}$. Let $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

$$Y = PU$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1e^{2x} - (1/4)c_2e^{-3x} \\ c_1e^{2x} + c_2e^{-3x} \end{bmatrix}$$
$$y_1 = c_1e^{2x} - (1/4)c_2e^{-3x}$$
$$y_2 = c_1e^{2x} + c_2e^{-3x}.$$

Now the initial conditions given $y_1(0) = 1,\ y_2(0) = 6$ then $c_1 = 2, c_2 = 4.$

Therefore, $y_1 = 2e^{2x} - e^{-3x}$

$$y_2 = 2e^{2x} + 4e^{-3x}.$$

Application to Quadratic Surfaces

An equation of the form

$$ax^2 + by^2 + cz^2 + 2dxy + 2fyz + gx + hy + iz + j = 0$$

where $a, b, \cdots, f$ are not all zero, is called a *quadratic equation* in x, y and $z.$ The expression

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$$

is called the *associated quadratic* form.

The quadratic equation can be written in the matrix form

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + j = 0$$
$$X^tAX + KX + j = 0$$

where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}, K = \begin{bmatrix} g & h & i \end{bmatrix}.$$

The symmetric matrix A is called the *matrix of the quadratic* form.

$$X^tAX = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz.$$

Theorem (5)

Let

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + gx + hy + iz + j = 0$$

be the equation of a quadric Q, and let

$$X^tAX = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$$

be the associated quadric form. Then the coordinate axes can be rotated so that the equation of Q in the $x'y'z'$-coordinate system has the form

$$\lambda_1x'^2 + \lambda_2y'^2 + \lambda_3z'^2 + g'x' + h'y' + i'z' + j = 0$$

where λ_1, λ_2 and λ_3 are the eigenvalues of A. The rotation can be accomplished by the substitution

$$X = PX'$$

where P orthogonally diagonalizes A and $\det(P) = 1.$

This theorem suggests the following procedure for removing the cross-product terms from a quadratic equation in x, y and $z.$

Example (3)

The quadric surface whose equation is

$$4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz - 3 = 0.$$

The matrix form of the equation is

$$X'AX - 3 = 0$$

where

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$
$$\det(\lambda I - A) = 0$$
$$(\lambda - 2)(\lambda - 8) = 0$$
$$\lambda = 2,\ \lambda = 8.$$

The eigenvalues of A are $\lambda = 2$ and $\lambda = 8$, and A is orthogonally diagonalized by the matrix

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

where the first two column vectors in P are eigenvectors corresponding to $\lambda = 2$ and the third column vector is an eigenvector corresponding to $\lambda = 8$.

Since $\det(P) = 1$, the orthogonal coordinate transformation

$$X = PX',$$

that is, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

is a rotation.

By substituting in $X'AX - 3 = 0$, yields

$$(PX')^t A(PX') - 3 = 0$$

$$(X')^t (P^t AP)X' - 3 = 0.$$

Since $P^t AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, we get

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} - 3 = 0$$

$$2x'^2 + 2y'^2 + 8z'^2 = 3$$

or

$$\frac{x'^2}{3/2} + \frac{y'^2}{3/2} + \frac{z'^2}{3/8} = 1$$

is the equation of an ellipsoid as shown in Figure 1.

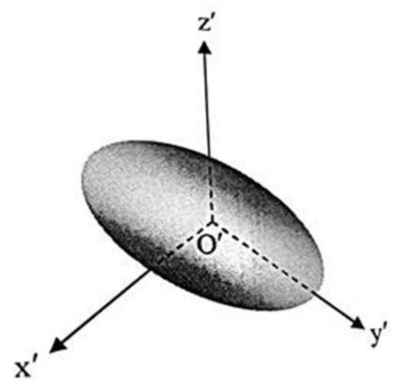

Figure 1 An ellipsoid

Conclusion

This research paper is limited to linear transformations on finite-dimensional vector spaces. Thus, V is denoted a finite-dimensional vector space over a field F. The algebra $A(V)$ has a unit element I and if $T \in A(V)$ then $\lambda \in F$ is the eigenvalue of T. By using eigenvalues, the corresponding eigenvectors of T are attained. Linear algebra can be applied to solve certain systems of differential equation with eigenvalues and eigenvectors.

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References

Anton, H., (1981). "*Elementary Linear Algebra*", Third Edition, John Wiley & Sons, New York.

Herstein, I.N., (1975). "*Topic In Algebra*", John Wiley & Sons, New York.

Khanna, V.K., and S.K., Bhambri., (1998)."*A Course In Abstract Algebra*", Second Rivised Edition, Vikas Publishing House Pvt Ltd, New Delhi.